9.5 Trigonometric Ratios

What *you should learn*

GOAL 1 Find the sine, the cosine, and the tangent of an acute angle.

GOAL 2 Use trigonometric ratios to solve **real-life** problems, such as estimating the height of a tree in **Example 6**.

Why *you should learn it*

▼ To solve **real-life** problems, such as in finding the height of a water slide in **Ex. 37**.

REAL LIFE

GOAL 1 FINDING TRIGONOMETRIC RATIOS

A **trigonometric ratio** is a ratio of the lengths of two sides of a right triangle. The word *trigonometry* is derived from the ancient Greek language and means measurement of triangles. The three basic trigonometric ratios are **sine**, **cosine**, and **tangent**, which are abbreviated as *sin*, *cos*, and *tan*, respectively.

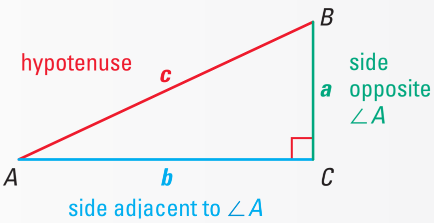
TRIGONOMETRIC RATIOS

Let $\triangle ABC$ be a right triangle. The sine, the cosine, and the tangent of the acute angle $\angle A$ are defined as follows.

$$\sin A = \frac{\text{side opposite } \angle A}{\text{hypotenuse}} = \frac{a}{c}$$

$$\cos A = \frac{\text{side adjacent to } \angle A}{\text{hypotenuse}} = \frac{b}{c}$$

$$\tan A = \frac{\text{side opposite } \angle A}{\text{side adjacent to } \angle A} = \frac{a}{b}$$



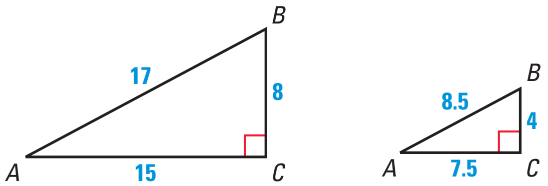
The value of a trigonometric ratio depends only on the measure of the acute angle, not on the particular right triangle that is used to compute the value.

EXAMPLE 1 *Finding Trigonometric Ratios*

Compare the sine, the cosine, and the tangent ratios for $\angle A$ in each triangle below.

SOLUTION

By the SSS Similarity Theorem, the triangles are similar. Their corresponding sides are in proportion, which implies that the trigonometric ratios for $\angle A$ in each triangle are the same.



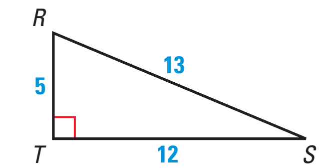
	Large triangle	Small triangle
$\sin A = \frac{\text{opposite}}{\text{hypotenuse}}$	$\frac{8}{17} \approx 0.4706$	$\frac{4}{8.5} \approx 0.4706$
$\cos A = \frac{\text{adjacent}}{\text{hypotenuse}}$	$\frac{15}{17} \approx 0.8824$	$\frac{7.5}{8.5} \approx 0.8824$
$\tan A = \frac{\text{opposite}}{\text{adjacent}}$	$\frac{8}{15} \approx 0.5333$	$\frac{4}{7.5} \approx 0.5333$

Trigonometric ratios are frequently expressed as decimal approximations.

EXAMPLE 2 Finding Trigonometric Ratios

Find the sine, the cosine, and the tangent of the indicated angle.

a. $\angle S$ **b.** $\angle R$



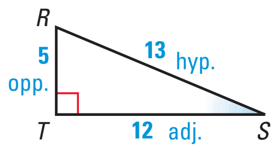
SOLUTION

a. The length of the hypotenuse is 13. For $\angle S$, the length of the opposite side is 5, and the length of the adjacent side is 12.

$$\sin S = \frac{\text{opp.}}{\text{hyp.}} = \frac{5}{13} \approx 0.3846$$

$$\cos S = \frac{\text{adj.}}{\text{hyp.}} = \frac{12}{13} \approx 0.9231$$

$$\tan S = \frac{\text{opp.}}{\text{adj.}} = \frac{5}{12} \approx 0.4167$$

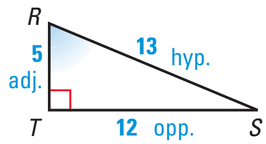


b. The length of the hypotenuse is 13. For $\angle R$, the length of the opposite side is 12, and the length of the adjacent side is 5.

$$\sin R = \frac{\text{opp.}}{\text{hyp.}} = \frac{12}{13} \approx 0.9231$$

$$\cos R = \frac{\text{adj.}}{\text{hyp.}} = \frac{5}{13} \approx 0.3846$$

$$\tan R = \frac{\text{opp.}}{\text{adj.}} = \frac{12}{5} = 2.4$$



. . . .

You can find trigonometric ratios for 30°, 45°, and 60° by applying what you know about special right triangles.

STUDENT HELP

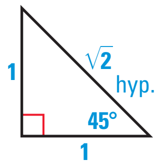
INTERNET **HOMEWORK HELP**
Visit our Web site www.mcdougallittell.com for extra examples.

EXAMPLE 3 Trigonometric Ratios for 45°

Find the sine, the cosine, and the tangent of 45°.

SOLUTION

Begin by sketching a 45°-45°-90° triangle. Because all such triangles are similar, you can make calculations simple by choosing 1 as the length of each leg. From Theorem 9.8 on page 551, it follows that the length of the hypotenuse is $\sqrt{2}$.



$$\sin 45° = \frac{\text{opp.}}{\text{hyp.}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \approx 0.7071$$

$$\cos 45° = \frac{\text{adj.}}{\text{hyp.}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \approx 0.7071$$

$$\tan 45° = \frac{\text{opp.}}{\text{adj.}} = \frac{1}{1} = 1$$

STUDENT HELP

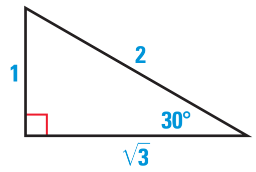
Study Tip
The expression sin 45° means the sine of an angle whose measure is 45°.

EXAMPLE 4 *Trigonometric Ratios for 30°*

Find the sine, the cosine, and the tangent of 30°.

SOLUTION

Begin by sketching a 30°-60°-90° triangle. To make the calculations simple, you can choose 1 as the length of the shorter leg. From Theorem 9.9 on page 551, it follows that the length of the longer leg is $\sqrt{3}$ and the length of the hypotenuse is 2.



$$\sin 30° = \frac{\text{opp.}}{\text{hyp.}} = \frac{1}{2} = 0.5$$

$$\cos 30° = \frac{\text{adj.}}{\text{hyp.}} = \frac{\sqrt{3}}{2} \approx 0.8660$$

$$\tan 30° = \frac{\text{opp.}}{\text{adj.}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \approx 0.5774$$

EXAMPLE 5 *Using a Calculator*

You can use a calculator to approximate the sine, the cosine, and the tangent of 74°. Make sure your calculator is in *degree mode.* The table shows some sample keystroke sequences accepted by most calculators.

Sample keystroke sequences	Sample calculator display	Rounded approximation
74 SIN or SIN 74 ENTER	0.961261695	0.9613
74 COS or COS 74 ENTER	0.275637355	0.2756
74 TAN or TAN 74 ENTER	3.487414444	3.4874

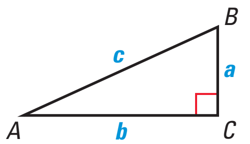
STUDENT HELP

Trig Table
For a table of trigonometric ratios, see p. 845.

If you look back at Examples 1–5, you will notice that the sine or the cosine of an acute angle is always less than 1. The reason is that these trigonometric ratios involve the ratio of a leg of a right triangle to the hypotenuse. The length of a leg of a right triangle is always less than the length of its hypotenuse, so the ratio of these lengths is always less than one.

Because the tangent of an acute angle involves the ratio of one leg to another leg, the tangent of an angle can be less than 1, equal to 1, or greater than 1.

TRIGONOMETRIC IDENTITIES A trigonometric identity is an equation involving trigonometric ratios that is true for all acute angles. You are asked to prove the following identities in Exercises 47 and 52:



$$(\sin A)^2 + (\cos A)^2 = 1$$

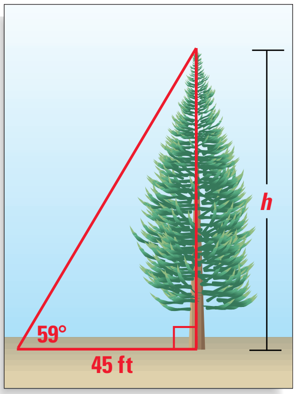
$$\tan A = \frac{\sin A}{\cos A}$$

GOAL 2 USING TRIGONOMETRIC RATIOS IN REAL LIFE

Suppose you stand and look up at a point in the distance, such as the top of the tree in Example 6. The angle that your line of sight makes with a line drawn horizontally is called the **angle of elevation**.

EXAMPLE 6 *Indirect Measurement*

FORESTRY You are measuring the height of a Sitka spruce tree in Alaska. You stand 45 feet from the base of the tree. You measure the angle of elevation from a point on the ground to the top of the tree to be 59°. To estimate the height of the tree, you can write a trigonometric ratio that involves the height h and the known length of 45 feet.



$$\tan 59^\circ = \frac{\text{opposite}}{\text{adjacent}} \qquad \textbf{Write ratio.}$$

$$\tan 59^\circ = \frac{h}{45} \qquad \textbf{Substitute.}$$

$$45 \tan 59^\circ = h \qquad \textbf{Multiply each side by 45.}$$

$$45(1.6643) \approx h \qquad \textbf{Use a calculator or table to find tan 59°.}$$

$$74.9 \approx h \qquad \textbf{Simplify.}$$

▶ The tree is about 75 feet tall.

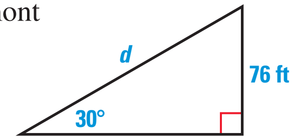
FOCUS ON CAREERS

FORESTRY Foresters manage and protect forests. Their work can involve measuring tree heights. Foresters can use an instrument called a *clinometer* to measure the angle of elevation from a point on the ground to the top of a tree.

CAREER LINK
www.mcdougallittell.com

EXAMPLE 7 *Estimating a Distance*

ESCALATORS The escalator at the Wilshire/Vermont Metro Rail Station in Los Angeles rises 76 feet at a 30° angle. To find the distance d a person travels on the escalator stairs, you can write a trigonometric ratio that involves the hypotenuse and the known leg length of 76 feet.



$$\sin 30^\circ = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \textbf{Write ratio for sine of 30°.}$$

$$\sin 30^\circ = \frac{76}{d} \qquad \textbf{Substitute.}$$

$$d \sin 30^\circ = 76 \qquad \textbf{Multiply each side by } \boldsymbol{d}\textbf{.}$$

$$d = \frac{76}{\sin 30^\circ} \qquad \textbf{Divide each side by sin 30°.}$$

$$d = \frac{76}{0.5} \qquad \textbf{Substitute 0.5 for sin 30°.}$$

$$d = 152 \qquad \textbf{Simplify.}$$

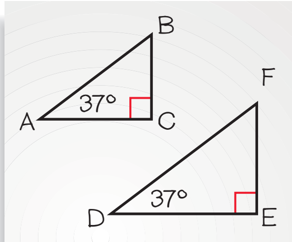
▶ A person travels 152 feet on the escalator stairs.

Guided Practice

Vocabulary Check ✓

Concept Check ✓

In Exercises 1 and 2, use the diagram at the right.

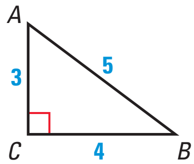


1. Use the diagram to explain what is meant by the *sine*, the *cosine*, and the *tangent* of $\angle A$.

2. **Error Analysis** A student says that $\sin D > \sin A$ because the side lengths of $\triangle DEF$ are greater than the side lengths of $\triangle ABC$. Explain why the student is incorrect.

Skill Check ✓

In Exercises 3–8, use the diagram shown at the right to find the trigonometric ratio.

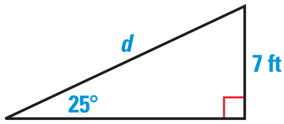


3. $\sin A$ **4.** $\cos A$

5. $\tan A$ **6.** $\sin B$

7. $\cos B$ **8.** $\tan B$

9. **Escalators** One early escalator built in 1896 rose at an angle of 25°. As shown in the diagram at the right, the vertical lift was 7 feet. Estimate the distance d a person traveled on this escalator.



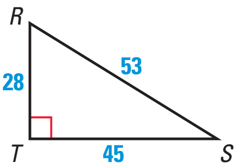
Practice and Applications

Student Help

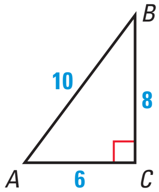
Extra Practice to help you master skills is on p. 820.

Finding Trigonometric Ratios **Find the sine, the cosine, and the tangent of the acute angles of the triangle. Express each value as a decimal rounded to four places.**

10.



11.



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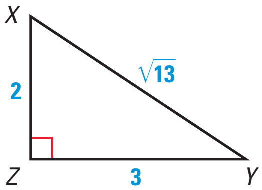
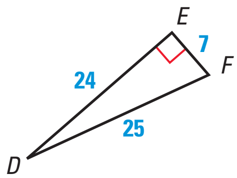


13.



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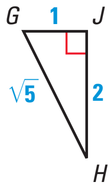

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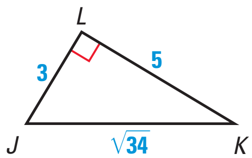


Student Help

Homework Help

Example 1: Exs. 10–15, 28–36
Example 2: Exs. 10–15, 28–36
Example 3: Exs. 34–36
Example 4: Exs. 34–36
Example 5: Exs. 16–27
Example 6: Exs. 37–42
Example 7: Exs. 37–42

Calculator **Use a calculator to approximate the given value to four decimal places.**

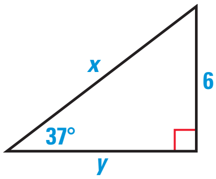
16. $\sin 48°$ **17.** $\cos 13°$ **18.** $\tan 81°$ **19.** $\sin 27°$

20. $\cos 70°$ **21.** $\tan 2°$ **22.** $\sin 78°$ **23.** $\cos 36°$

24. $\tan 23°$ **25.** $\cos 63°$ **26.** $\sin 56°$ **27.** $\tan 66°$

USING TRIGONOMETRIC RATIOS **Find the value of each variable. Round decimals to the nearest tenth.**

28.



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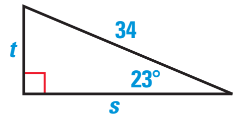

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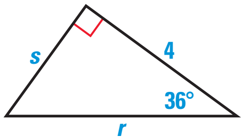

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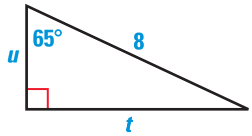

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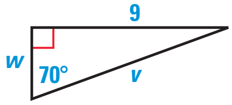
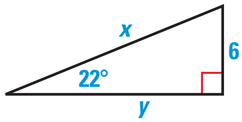
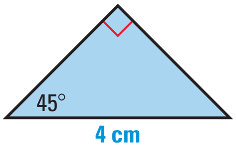


33.



FINDING AREA **Find the area of the triangle. Round decimals to the nearest tenth.**

34.



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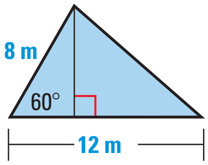

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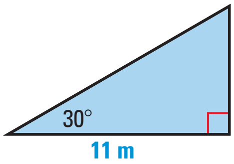
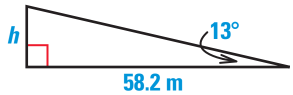


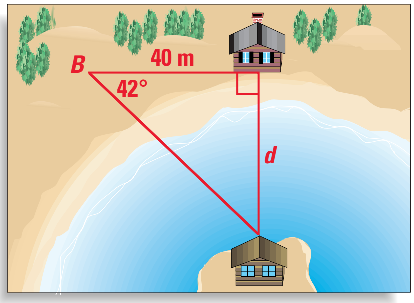
FOCUS ON APPLICATIONS

REAL LIFE **WATER SLIDE** Even though riders on a water slide may travel at only 20 miles per hour, the curves on the slide can make riders feel as though they are traveling much faster.

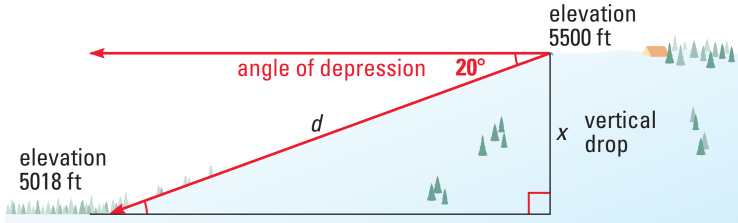
37. **WATER SLIDE** The angle of elevation from the base to the top of a waterslide is about 13°. The slide extends horizontally about 58.2 meters. Estimate the height h of the slide.



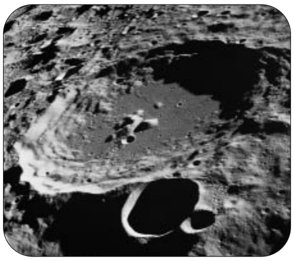
38. **SURVEYING** To find the distance d from a house on shore to a house on an island, a surveyor measures from the house on shore to point B, as shown in the diagram. An instrument called a *transit* is used to find the measure of $\angle B$. Estimate the distance d.



39. **SKI SLOPE** Suppose you stand at the top of a ski slope and look down at the bottom. The angle that your line of sight makes with a line drawn horizontally is called the *angle of depression*, as shown below. The *vertical drop* is the difference in the elevations of the top and the bottom of the slope. Find the vertical drop x of the slope in the diagram. Then estimate the distance d a person skiing would travel on this slope.



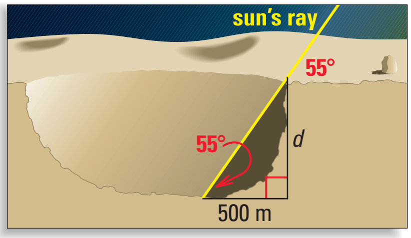
FOCUS ON APPLICATIONS



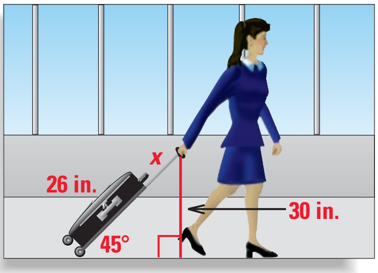
REAL LIFE

LUNAR CRATERS Because the moon has no atmosphere to protect it from being hit by meteorites, its surface is pitted with craters. There is no wind, so a crater can remain undisturbed for millions of years—unless another meteorite crashes into it.

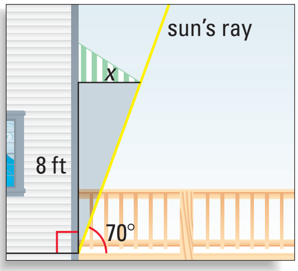
40. **SCIENCE CONNECTION** Scientists can measure the depths of craters on the moon by looking at photos of shadows. The length of the shadow cast by the edge of a crater is about 500 meters. The sun's angle of elevation is 55°. Estimate the depth d of the crater.



41. **LUGGAGE DESIGN** Some luggage pieces have wheels and a handle so that the luggage can be pulled along the ground. Suppose a person's hand is about 30 inches from the floor. About how long should the handle be on the suitcase shown so that it can roll at a comfortable angle of 45° with the floor?

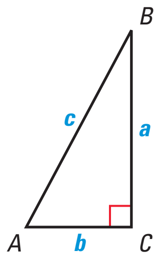


42. **BUYING AN AWNING** Your family room has a sliding-glass door with a southern exposure. You want to buy an awning for the door that will be just long enough to keep the sun out when it is at its highest point in the sky. The angle of elevation of the sun at this point is 70°, and the height of the door is 8 feet. About how far should the overhang extend?



CRITICAL THINKING **In Exercises 43 and 44, use the diagram.**

43. Write expressions for the sine, the cosine, and the tangent of each acute angle in the triangle.

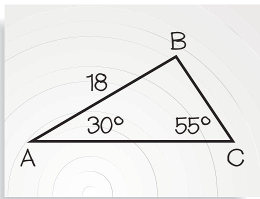


44. *Writing* Use your results from Exercise 43 to explain how the tangent of one acute angle of a right triangle is related to the tangent of the other acute angle. How are the sine and the cosine of one acute angle of a right triangle related to the sine and the cosine of the other acute angle?

45. **TECHNOLOGY** Use geometry software to construct a right triangle. Use your triangle to explore and answer the questions below. Explain your procedure.

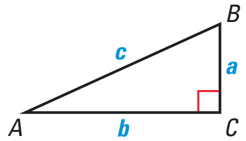
- For what angle measure is the tangent of an acute angle equal to 1?
- For what angle measures is the tangent of an acute angle greater than 1?
- For what angle measures is the tangent of an acute angle less than 1?

46. **ERROR ANALYSIS** To find the length of $\overline{BC}$ in the diagram at the right, a student writes $\tan 55° = \frac{18}{BC}$. What mistake is the student making? Show how the student can find BC. (*Hint:* Begin by drawing an altitude from B to $\overline{AC}$.)



47. **PROOF** Use the diagram of $\triangle ABC$. Complete the proof of the trigonometric identity below.

$$(\sin A)^2 + (\cos A)^2 = 1$$



GIVEN ▶ $\sin A = \frac{a}{c}, \cos A = \frac{b}{c}$

PROVE ▶ $(\sin A)^2 + (\cos A)^2 = 1$

Statements	Reasons
1. $\sin A = \frac{a}{c}, \cos A = \frac{b}{c}$	**1.** ?
2. $a^2 + b^2 = c^2$	**2.** ?
3. $\frac{a^2}{c^2} + \frac{b^2}{c^2} = 1$	**3.** ?
4. $\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1$	**4.** A property of exponents
5. $(\sin A)^2 + (\cos A)^2 = 1$	**5.** ?

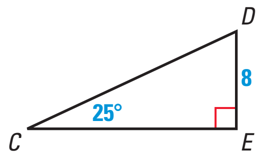
DEMONSTRATING A FORMULA **Show that $(\sin A)^2 + (\cos A)^2 = 1$ for the given angle measure.**

48. $m\angle A = 30°$ **49.** $m\angle A = 45°$ **50.** $m\angle A = 60°$ **51.** $m\angle A = 13°$

52. **PROOF** Use the diagram in Exercise 47. Write a two-column proof of the following trigonometric identity: $\tan A = \frac{\sin A}{\cos A}$.

Test Preparation

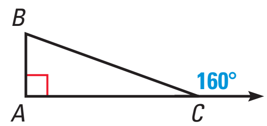
53. **MULTIPLE CHOICE** Use the diagram at the right. Find CD.



Ⓐ $8 \cos 25°$ Ⓑ $8 \sin 25°$ Ⓒ $8 \tan 25°$

Ⓓ $\frac{8}{\sin 25°}$ Ⓔ $\frac{8}{\cos 25°}$

54. **MULTIPLE CHOICE** Use the diagram at the right. Which expression is *not* equivalent to AC?

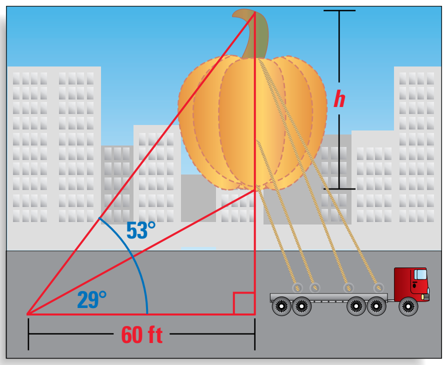


Ⓐ $BC \sin 70°$ Ⓑ $BC \cos 20°$ Ⓒ $\frac{BC}{\tan 20°}$

Ⓓ $\frac{BA}{\tan 20°}$ Ⓔ $BA \tan 70°$

★ Challenge

55. **PARADE** You are at a parade looking up at a large balloon floating directly above the street. You are 60 feet from a point on the street directly beneath the balloon. To see the top of the balloon, you look up at an angle of 53°. To see the bottom of the balloon, you look up at an angle of 29°. Estimate the height h of the balloon to the nearest foot.



EXTRA CHALLENGE
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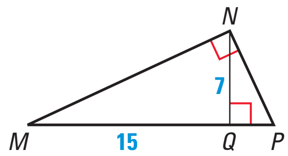
MIXED REVIEW

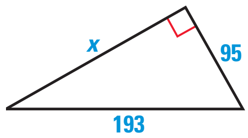
56. SKETCHING A DILATION $\triangle PQR$ is mapped onto $\triangle P'Q'R'$ by a dilation. In $\triangle PQR$, $PQ = 3$, $QR = 5$, and $PR = 4$. In $\triangle P'Q'R'$, $P'Q' = 6$. Sketch the dilation, identify it as a reduction or an enlargement, and find the scale factor. Then find the length of $Q'R'$ and $P'R'$. (Review 8.7)

57. FINDING LENGTHS Write similarity statements for the three similar triangles in the diagram. Then find QP and NP. Round decimals to the nearest tenth. (Review 9.1)

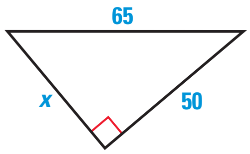


PYTHAGOREAN THEOREM Find the unknown side length. Simplify answers that are radicals. Tell whether the side lengths form a Pythagorean triple. (Review 9.2 for 9.6)

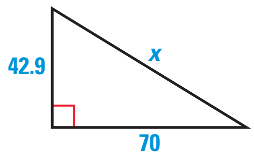
58.



59.



60.



QUIZ 2

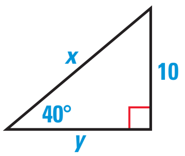
Self-Test for Lessons 9.4 and 9.5

Sketch the figure that is described. Then find the requested information. Round decimals to the nearest tenth. (Lesson 9.4)

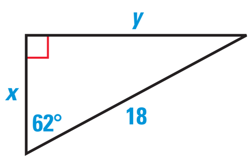
1. The side length of an equilateral triangle is 4 meters. Find the length of an altitude of the triangle.
2. The perimeter of a square is 16 inches. Find the length of a diagonal.
3. The side length of an equilateral triangle is 3 inches. Find the area of the triangle.

Find the value of each variable. Round decimals to the nearest tenth. (Lesson 9.5)

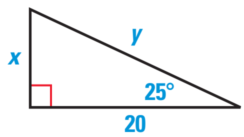
4.



5.



6.



7. HOT-AIR BALLOON The ground crew for a hot-air balloon can see the balloon in the sky at an angle of elevation of 11°. The pilot radios to the crew that the hot-air balloon is 950 feet above the ground. Estimate the horizontal distance d of the hot-air balloon from the ground crew. (Lesson 9.5)